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Mathematical Analysis and Grammatical Generation of Design Instances of Murcutt's Domestic Architecture

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Abstract: This paper presents a mathematical approach to analysing and generating design instances within a specific language of architectural design. The research develops a Shape Grammar allowing for the analysis of ten domestic designs by Glenn Murcutt and their generation. The Shape Grammar, which consists of eleven rule sets, starts by generating pavilions and ends with a termination rule. After describing the Shape Grammar the paper uses mathematics to directly measure and compare the rules, introducing the use of a Normalised Distance Graph (NDS), to capture the transition sequences of their application through rule transition paths. The results of this paper suggest that this approach can be used to clearly characterise design instances as well as to effectively create new designs in an architectural style. The mathematical approach is widely applicable to other grammatical studies in the architectural and design domains.

Keywords: Shape Grammar; mathematical analysis; design generation; Glenn Murcutt.

1. Introduction

Computational approaches to architectural analysis and generation have been developed and tested since the 1960s. One of the most notable theories from this field is "Shape Grammar", whose foundation has often been linked to Stiny and Gips' 1972 seminal article. The typical Shape Grammar approach treats architecture as an amalgam of shapes, examining the logical relationships between various sub-shapes that make up the two or three-dimensional forms of a building. The Shape Grammar approach specifies a set of rules delineating how a design can be composed from shapes. It starts with an initial shape and then proceeds iteratively by applying a set of rules (each of which specify a particular operation or a set of operations) to that shape until an end-state is reached. In this approach, design is assumed to be a rigorous and rational process. Such a conceptual understanding about the design process is central to most grammatical studies and this reasoning allows researchers to rigorously capture possible processes for generating a language of design (Economou, 2000; Knight, 2003). It is through this generative aspect that the selected architectural subjects are analysed and understood. The new knowledge of design that is developed in this way can then be tested by generating new designs that capture the characteristics of the original architecture, through the application of the same rule sets.

Although Shape Grammar theory has proven to be effective for architectural analysis, with many well-known applications from the field over the past four decades, the typical outcomes are largely descriptive including the definitions of the shapes, rules and sequences of the rule application. This paper proposes an alternative approach which mathematically evaluates the grammatical applications of certain rule sets to develop a language of architectural design (an architectural style). The mathematical measurements have a dual focus on normalised distance and transition probability.

This mathematical approach to the analysis and generation of architecture is demonstrated in the paper using ten of Pritzker-prize-winning architect Glenn Murcutt's rural houses. The paper starts by developing a Shape Grammar for the design analysis of the selected Murcutt's Houses. After describing the Shape Grammar the paper applies a range of mathematical means to directly measure and compare the rules and the sequences of their applications as well as the resultant designs. The mathematical approach also enables different designs and their generation processes to be compared and discussed in this paper.

2. A Shape Grammar on Murcutt's domestic buildings

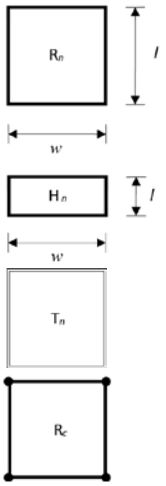
A Shape Grammar study specifies a set of rules delineating how designs can be composed by applying various operations to shapes. Most research in the field focuses on two dimensional (2D) shapes, typically floor plans (Stiny and Mitchell, 1978; Cagdas, 1996), on which the Murcutt Shape Grammar developed in this paper is also based. Logical structures are common in Shape Grammar research that often selectively adopts sequential structures comprising possible design steps. For example, Stiny and Mitchell (1978) use eight steps for several of their projects, whilst Hanson and Radford (1986) use 12 steps to generate their design instances. The Shape Grammar developed in this paper for Murcutt's rural houses consists of 11 rule sets in four phrases (see Table 1) which starts by generating pavilions and ends with a termination rule. The conceptual foundation for the generation of design instances typically involves both simplifying the properties of the shapes used in architecture and decomposing them using a modular system.

Table 1: Rule sets of a Shape Grammar for Murcutt's domestic buildings.

Stage	Rule set
Phrase 1	1.x. Generating pavilions
	2.x. Generating basic modules
Phrase 2	3.x. Configuring a core unit
	4.x. Configuring public zones (e.g. living room, dining room, kitchen, and function room)
	5.x. Configuring private zones (e.g. bedroom and studio)
	6.x. Configuring transition zones e.g. (veranda and court)
	7.x. Configuring hall units
	8.x. Configuring a garage
Phrase 3	9.x. Defining a main entrance
	10.x. Defining sub entrances
Phrase 4	11. Termination

The first phrase includes two rule sets to initially generate pavilions and basic modules. There are also two sub-rule sets to generate pavilions in the first rule set. Considering site contexts and design strategies, the Shape Grammar can be used to select a pavilion type from either a single-pavilion shape or a two-pavilion shape. Thus, Rule 1.1 generates a one-pavilion shape, while Rule 1.2 develops a two-pavilion shape, including two space units and a hall unit.

The second rule set generates basic modules (structural or functional bays) to standardise and delineate enclosed spaces. The bays can be simplified using a rectangular grid, which conforms to the repetitive column layout found in Murcutt's domestic architecture. While the dimensions of the enclosed spaces may vary in each of Murcutt's designs, the column lay-out is an essential characteristic of each house. For example, the Marie Short House consists of two types of modules (room-type and hall-type). The exact dimensions of modules are not considered in the grammar because of the need to simplify the number of variables used and the dimensions depend on a variety of design contexts that are not part of the grammar developed at the conceptual design stage. Instead, it considers Murcutt's plan layouts with four basic units with simplified constants (e.g. *a* for width, *b* for length) in Figure 1, for the mapping of space: room, hall, transit, and core unit – to develop the rules of the grammar.



Room unit	A room unit is a basic convex space with a defined function. (e.g. living room, bedroom, and kitchen). Generally, the width (*w*) is *a* or 2*a* and the length (*l*) is *b*. 'R_n' is a marker representing a basic room unit, but 'A_n' can also be used for indicating an attached room unit.
Hall Unit	A hall unit is assigned as a hall. The length of an attached hall unit is generally *c*.
Transit unit	A transit unit is assigned mainly for a veranda.
Core unit	A core unit using a marker, denoted as 'R_c', includes a main entrance. The width is *a* or 2*a* and the length is *b* or *b*+*c*. The core unit is distinguished by assigning the symbol (●) at its corners.

Figure 1: Four basic units for the grammar.

There are three rules in the second rule set – Rule2.1 generating modules consisting of a shape space; Rule2.2 generating a series of modules consisting of a room (R_n) and a hall unit module (H_n) and an attached room unit module (A_n); Rule2.3 generating a series of modules consisting of a space (S_n) and a hall unit module (H_n) – as described in Figure 2. The second phrase consists of six rule sets which configure walls according to spatial functions. The rule set 3 configures a core unit, as a starting point for the remaining configuration, which includes a main entrance in the ninth step of the grammar. Rule3.1 develops a room shape module into a defined core unit (R_C), while two space shape modules are changed to a core unit, which results in a double-size room unit (2a) by Rule3.2. Rule3.3 considers a room unit and a hall unit as a core unit. Through Rule3.4 the composition of a room unit, a hall unit, and an attached space unit becomes a core unit (see Figure 2).

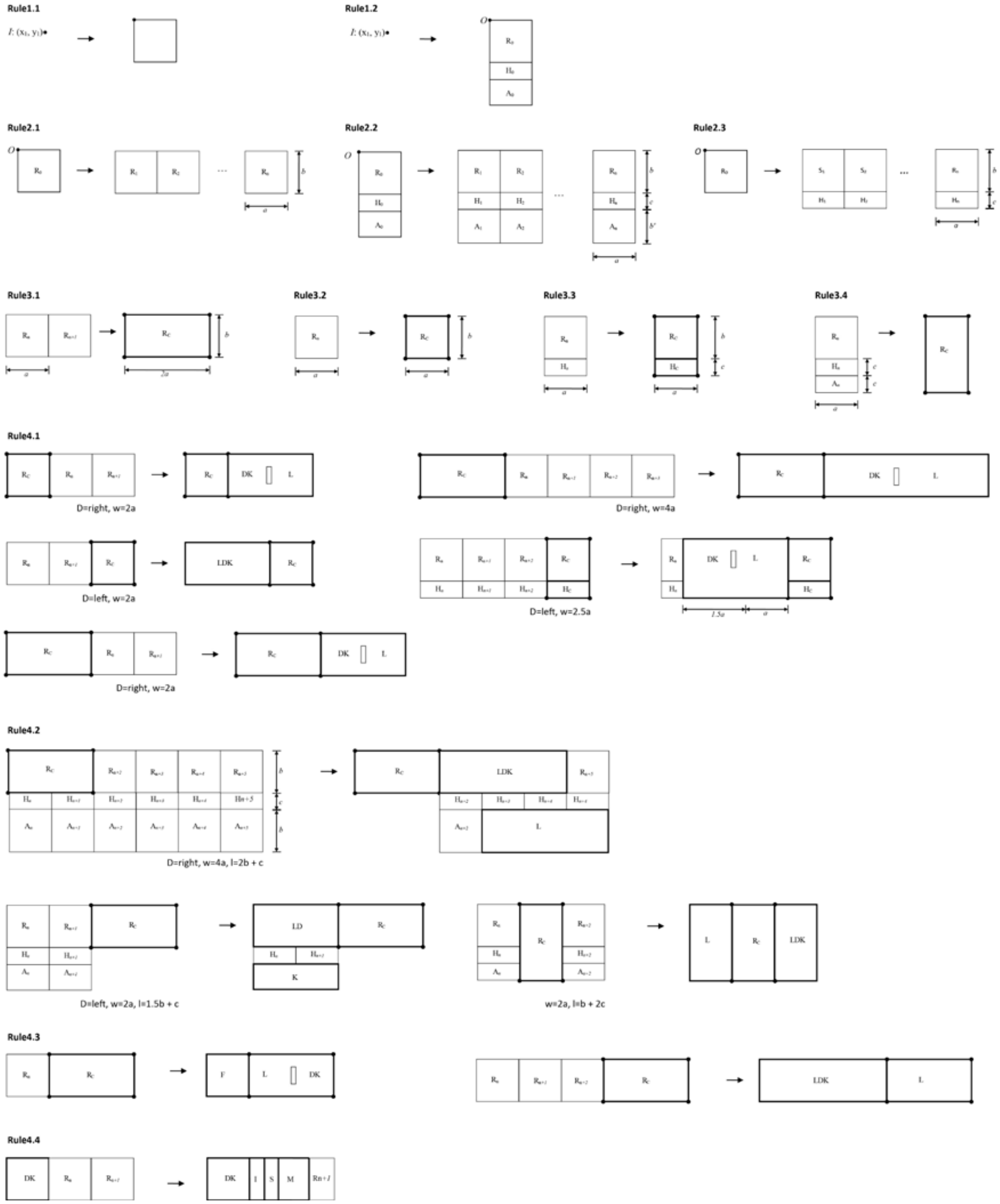


Figure 2: Rule sets 1-4 of Murcutt's Shape Grammar (other sets omitted due to space limit).

Rule set 4 configures public zones which consist of living room (L), dining room (d), Kitchen (K), LD, LDK, DK, etc. There are four rules: Rule4.1 generating one public zone (LDK, LD+K) from the side of a core unit; Rule4.2 generating separated public zones; Rule4.3 generating public zones including a core unit; Rule4.4 generating other public zones such as a music room. Rule set 5 configures private zones, e.g., bedroom and studio. There are five rules: Rule5.1 generating one bedroom from the side of a public zone; Rule5.2 generating two bedrooms from the side of a public zone; Rule5.3 generating a bedroom from the side of a core unit; Rule5.4 generating two bedrooms from the side of a core unit; Rule5.5 changing a core unit into a bedroom. Rule set 6 configures transition zones such as verandas. The first rule of this set is skipping and going to the next rule set, while Rule6.2 develops verandas and Rule6.3 generates a court. Rule set 7 configures hall units. Rule7.1 develops hall units into a hall unit, while Rule7.2 changes part of a core unit into a hall unit. Rule set 8 defines a garage unit through Rule8.1 or skips and goes to Rule set 9 by Rule8.2. The third phrase consists of two rule sets. Defining a main entrance involves three rules: Rule9.1 defining a main entrance in a hall space; Rule9.2 defining a main entrance in transition zones; Rule9.3 defining a main entrance in public zones. The process of defining sub Entrances can be triggered more than once. There are five rules: Rule10.1 defining a sub-entrance in a garage unit; Rule10.2 defining a sub-entrance in a hall unit; Rule10.3 defining a sub-entrance in public zones; Rule10.4 defining a sub-entrance in private zones; Rule10.5 skipping and going to ‘Termination’. The final phrase (rule set 11) terminates the generation process.

3. Mathematical analysis on the application of the grammar

3.1. Frequency of applied rule sets

An application of the Shape Grammar is demonstrated using ten houses by Glenn Murcutt (built between 1975 and 2005) syntactically examined in Ostwald (2011a; 2011b). The first part of this section is concerned with the tendency of the applied rules in the cases and the second with an alternative way of characterising each case through a mathematical abstraction using the frequency of applied rule sets, the so called ‘normalised distance’.

Table 2: Rules applied to generate the ten cases.

Rule set	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8	Case 9	Case 10
1.x	1.1	1.1	1.1	1.1	1.1	1.1	1.2	1.2	1.2	1.2
2.x	2.1	2.1	2.1	2.1	2.3	2.3	2.2	2.2	2.2	2.2
3.x	3.1	3.1	3.2	3.2	3.3	3.1	3.4	3.1	3.1	3.1
4.x	4.1	4.1	4.1	4.1	4.1	4.2, 4.4	4.2	4.3	4.3	4.2
5.x	5.3, 5.5	5.4	5.1, 5.4	5.4	5.1, 5.3	5.2	5.1, 5.2	5.1, 5.2	5.2	5.1, 5.3
6.x	6.1	6.2(2)*	6.1	6.1	6.1	6.1	6.3	6.2(2)	6.2	6.1
7.x	7.2	7.2	7.2	7.2	7.2	7.1(2)	7.1	7.1(3)	7.1(2)	7.1
8.x	8.2	8.2	8.1	8.1	8.2	8.1	8.1	8.2	8.2	8.1
9.x	9.1	9.1	9.1	9.1	9.1	9.3	9.2	9.2	9.2	9.3
10.x	10.5	10.5	10.1, 10.3, 10.4	10.1, 10.3	10.2	10.1, 10.2, 10.3	10.1, 10.4	10.1 10.2(2)	10.2, 10.3	10.1, 10.2

* The number inside parenthesis indicates the number of times the rule is applied.

Table 2 illustrates the rules applied to generate the ten cases. For example, in order to generate the first case, the Shape Grammar uses the set of rules, 1.1, 2.1, 3.1, 4.1, 5.3, 5.5, 6.1, 7.2, 8.2, 9.1 and 10.5. In cases 1 to 6, Rule1.1 is applied in the first rule set to generate single-pavilion shapes, while Rule1.2 is used for the two-pavilion shapes in cases 7 to 10. This table provides the information required to conduct the mathematical analysis on the application of the grammar.

The frequency of the applied rules can be recorded within the set of the ten designs. This information indicates a tendency to select a particular rule or pattern. The tendency of each rule forms a typical 'Murcutt-esque' language or style. In order to investigate the tendency, we firstly sorted the applied rules at each step of the grammar. Secondly, the frequency of each applied rule was calculated. Finally, the most frequently applied rule was located in the first sub rule (x.1) and the next frequently applied rule was set to the second sub rule (x.2). This means the rule close to the first sub rule in each rule set, can more likely generate an archetype in the language of design because it is most frequently applied (see Table 3).

Table 3: Frequencies of each applied rule.

Rule set	x.1	x.2	x.3	x.4	x.5	Mean	SD
Rule1	6	4	-	-	-	5.00	1.41
Rule2	4	4	2	-	-	3.33	1.15
Rule3	5	3	1	1	-	2.50	1.91
Rule4	5	3	2	1	-	2.50	1.71
Rule5	5	4	3	3	1	3.20	1.48
Rule6	6	5	1	-	-	4.00	2.65
Rule7	9	5	-	-	-	6.50	3.54
Rule8	5	5	-	-	-	5.00	0.00
Rule9	5	3	2	-	-	3.33	1.25
Rule10	6	6	4	2	2	4.00	2.00

3.2. Normalised Distance Graph (NDG)

Based on the data in Table 3, this paper introduces an alternative approach to analysing the application of the grammar and to visualising the characteristics of each case in terms of the grammatical design process. Table 4 determines the normalised distances that are calculated using applied rule's normalised frequency in relation to the rule that is most frequently applied. Each normalised value indicates the standardised frequency based on each mean frequency. Rule set space is denoted by $Rule_x$ = (x=1 to 10). $Rule_x$= {$Rule_{x.y}$: y =1, 2, 3…k}, k = max number of sub rule y in the rule set, $Rule_x$. The normalised frequency of one of the rules $Rule_{x.y} \in Rule_x$ is:

$$F_{normalised}(Rule_{x.y}) = F'(Rule_{x.y}) = \frac{F(Rule_{x.y}) - \overline{F(Rule_x)}}{SD} \qquad (1)$$

Where:

F is the frequency of each rule being applied in the ten houses, SD is the standard deviation of the frequencies of $Rule_x$. Each rule's normalised distance is then calculated by the absolute value of each normalised frequency subtracted by the normalised value of the most frequently applied rule. The sub rules of each rule set are already ordered by the application frequency in Murcutt's ten houses (see Table

3). That is, sub rule $Rule_{x.1}$ is the most frequently applied rule in each rule set. Thus, the Normalised Distance (ND) of the frequency of one of the rules $Rule_{x.y} \in Rule_x$ is:

$$\text{ND}\left(Rule_{x.y}\right) = \left| F'(Rule_{x.y}) - F'(Rule_{x.1}) \right| \quad (2)$$

Table 4: Normalised Distance of each applied rule.

Rule set	x.1	x.2	x.3	x.4	x.5
Rule1	0.00	1.41	-	-	-
Rule2	0.00	0.00	1.73	-	-
Rule3	0.00	1.04	2.09	2.09	-
Rule4	0.00	1.17	1.76	2.34	-
Rule5	0.00	0.67	1.35	1.35	2.70
Rule6	0.00	0.38	1.89	-	-
Rule7	0.00	1.41	-	-	-
Rule8	0.00	0.00	-	-	-
Rule9	0.00	1.60	2.41	-	-
Rule10	0.00	0.00	1.00	2.00	-

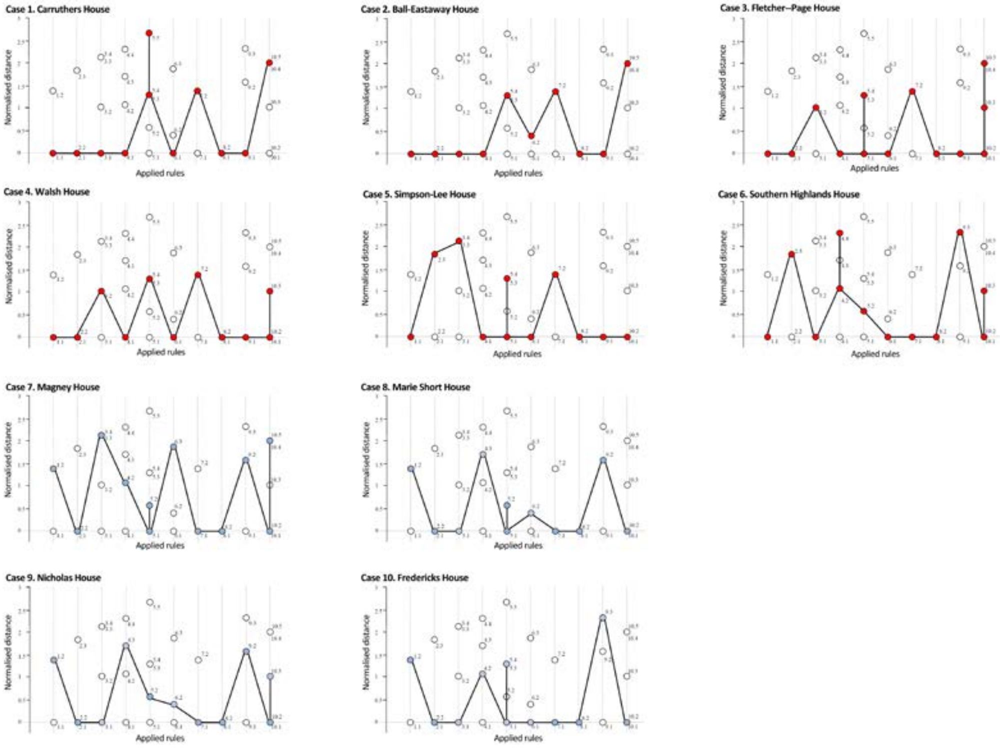


Figure 3: NDGs of applied rule sets to generate ten Murcutt's rural houses.

Table 4 calculates the ND of each applied rule. Since the first sub rule is the most frequently applied rule in the rule set, its ND is always zero. The farthest sub rule is Rule5.5, whose distance is 2.70. Thus, the NDs can easily demonstrate the disparity of each rule from a typical rule in the set. This also enables effective visualisation of the grammar application to generate design instances. Figure 3 illustrates the Normalised Distance Graphs (NDGs) of applied rule sets to generate the ten cases. The NDGs show that the third case may most closely capture the Murcutt style because of the lowest overall ND values. The NDGs also reveal that cases 1 and 2 would be similar designs due to the similarity of the two graphs.

4. Generation of design instances

4.1. Probability of applied rules

As shown in the previous section, the application of the first sub rule in each rule set, as the dominant rule, may generate a typical instance in Murcutt's domestic buildings. Thus, the probability of the applied rule enables us to mathematically investigate the generation of design instances.

Rule set space is denoted by $Rule_x$ = (x=1 to 10). $Rule_x$= {$Rule_{x.a}$: a =1, 2, 3…k}, k = max number of sub rule y in the rule set R_x. The probability (P) of one of the rules $Rule_{x.a}$ by the frequency of all rules in each rule set occurring in the ten houses is:

$$P\left(Rule_{x.a}\right) = \frac{F(Rule_{x.a})}{\sum_1^k F(Rule_{x.k})} \qquad (3)$$

Where:

F is the frequency of each rule occurring in the ten houses. Table 5 describes the probability of the applied rules to generate a Murcutt houses. For example, the probability of the application of Rule3.1 is 5/10, 50%, while the one of Rule3.2 is 3/10, 30%. The probability of Rule3.1 means that a room shape module is developed into a defined core unit in one out of two cases. The multiplication of the probabilities of applied rules also represents the probability of the generation of such a design instance developed by all the rules. This following section focuses on new design generation using rule transition paths based on the probability.

Table 5: Probability of applied rules (F = frequency).

Rule set	Sub rule1	Sub rule2	Sub rule3	Sub rule4	Sub rule5	Total F.
1.x	6/10	4/10	-	-	-	10
2.x	4/10	4/10	2/10	-	-	10
3.x	5/10	3/10	1/10	1/10	-	10
4.x	5/11	3/11	2/11	1/11	-	11
5.x	5/16	4/16	3/16	3/16	1/16	16
6.x	6/12	5/12	1/12	-	-	12
7.x	9/14	5/14	-	-	-	14
8.x	5/10	5/10	-	-	-	10
9.x	5/10	3/10	2/10	-	-	10
10.x	6/20	6/20	4/20	2/20	2/20	20

4.2. Design generation using rule transition paths

To generate new design instances in the Murcutt language, the research considers the transition probability. The transition probability (*T*) is the probability of transitioning from one rule to the following rule in the sequential step. Thus, generating a certain design can be a response to the given contexts or constraints of the grammar. If a rule is currently in a rule set (state) $R_{x.a}$, then it moves to Rule set $R_{y.b}$ at the next step with a probability denoted by $T_{x.a \rightarrow y.b}$ where $x, y, a, b > 1$.

$T_{x.a \rightarrow y.b}$ enables us to select more appropriate rules at each rule set in terms of the grammatical application. In addition, adopting a more possible rule at each sequential rule set arguably generates more typical designs in terms of Murcutt's design styles based on the ten houses. From the applied rules to the following rules, the transition probabilities are determined. For example, the transition probability for Rules set 1 and 2 (R_1 and R_2) is:

$$T_{x.a \rightarrow y.b} = \begin{array}{c} \\ R_{1.1} \\ R_{1.2} \end{array} \begin{array}{c} \begin{array}{ccc} R_{2.1} & R_{2.2} & R_{2.3} \end{array} \\ \begin{pmatrix} 4/6 & 0 & 2/6 \\ 0 & 4/4 & 0 \end{pmatrix} \end{array}$$

Generally, both the transition probability (*T*) and the probability (*P*) are used for the grammar application. If there is more than one rule that has the maximum transition probability at each generation stage, the lower numbered sub set rule – the higher probability of the applied rule in Table 5 – is triggered. For example, the probabilities of the generation of two pavilions are respectively 6/10, 4/10. Thus, the application generally selects one pavilion (Rule 1.1). In order to generate the following modules (Rule2.x), if the application chooses the dominant rule, Rule1.1, then it is followed by Rule2.1. Based on these transition probabilities, the research develops rule transition paths considering the maximum value of each transition probability (see Figure 4).

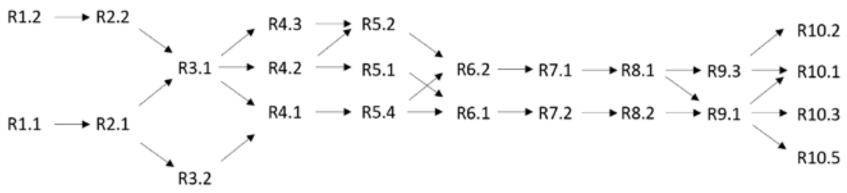


Figure 4: Rule transition paths considering the maximum value of each transition probability.

Through the rule transition paths, new design instances can be generated (see Figure 5). For example, "Rule 1.1 → Rule 2.1→ Rule 3.1 → Rule 4.1 → Rule 5.3 → Rule 6.1→ Rule 7.2 → Rule 8.2 → Rule 9.1" can be applied as a typical instance, even if Rule 10.1 is skipped due to the missing garage. Since there is more than one rule that has the maximum transition probability at each generation stage, several design alternatives can effectively be generated through these transition paths.

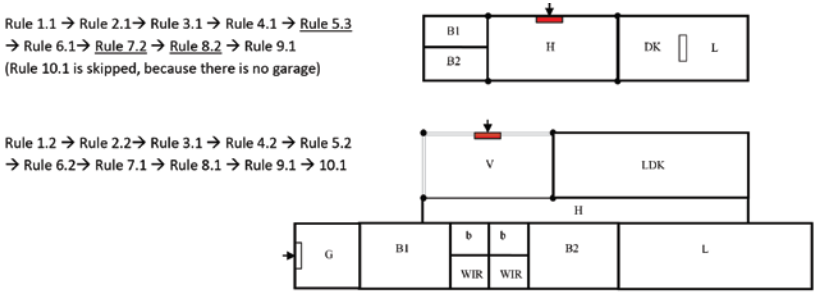


Figure 5: Two of the possible design instances generated by the rule transition paths.

5. Conclusion

The mathematical analysis and generation of design instances presented in this paper is a general method that can be applied to the other Shape Grammar studies. It articulates the measurement of a grammar in terms of the frequency of the applied rules and the categorisation of rule sets that allow for the exploration of a particular architectural language or style. Thus, the research contributes to a better understanding of the grammatical design process implicit in the architecture that is being analysed.

The method also supports both quantitative and qualitative examination of the grammar's application. The development of NDGs allows us to visually investigate the rule applications as well as to numerically describe the difference (through ND) of each applied rule to the dominant rule. Together they enable the investigation of the similarity and disparity between design instances as well as the differentiation of a 'typical' design instance from an 'abnormal' one. The results suggest that this mathematical approach can be used to systematically characterise or compare design instances as well as to effectively create new designs in an architectural style. The rule transition paths illustrated in Figure 4 have sequential steps that have been simplified for the demonstration purpose, nevertheless it provides an effective method for revealing a logical analogue of the design processes which can be used to support academics and professionals using the Shape Grammar.

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